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THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. VI. MAY, 1899. No. 5.

ON SYMMETRIC FUNCTIONS.

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[Continued from April Number.]

5. FORMULAS FOR THE FOUR KINDS OF REDUCTION.

We will place here together the four formulas of reduction. They are:

$$\begin{pmatrix} 0^{\lambda_0} & 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^m & \varkappa_1 \;\varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix} = \begin{pmatrix} 0^{\lambda_0-\lambda} & 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^{\kappa_1} & \varkappa_1 \;\varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix}, \quad m-\lambda=\kappa_1,$$

$$\begin{pmatrix} 0^{\lambda_0} & 1^{\lambda_1} \ldots\ldots n^{\lambda_n} \\ 0^m & \varkappa_1 \;\varkappa_2 \ldots\ldots \varkappa_n \end{pmatrix} = (-1)^{n\kappa_n} \begin{pmatrix} 0^{\lambda_0} & 1^{\lambda_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_n-\kappa_n} \\ 0^{m-\kappa_n} & (\varkappa_1-\varkappa_n)(\varkappa_2-\varkappa_n)\ldots(\varkappa_{n-1}-\varkappa_n) \end{pmatrix},$$

$$\begin{pmatrix} 0^{\lambda_0} & 1^{\lambda_1} \ldots r (r+1)^0 \ldots n^0 \\ 0^m & \varkappa_1 \;\varkappa_2 \ldots\ldots\ldots \varkappa_r 0^{n-r} \end{pmatrix} = \begin{pmatrix} 0^{\lambda_0} & 1^{\lambda_1} \ldots\ldots r^{\lambda_r} \\ 0^m & \varkappa_1 \;\varkappa_2 \ldots \varkappa_r \end{pmatrix},$$

$$\begin{pmatrix} r^{\lambda_r} & (r+1)^{\lambda_{r+1}} \ldots n^{\lambda_n} \\ 0^m & \varkappa_1 \;\varkappa_2 \ldots\ldots\ldots \varkappa_n \end{pmatrix} = (-1)^{\kappa_1 r} \begin{pmatrix} 0^{\lambda_r} & 1^{\lambda_{r+1}} \ldots (n-r)^{\lambda_n} \\ 0^{\kappa_1} & \varkappa_{r+1} \;\varkappa_{r+2} \ldots\ldots \varkappa_n \end{pmatrix}, \quad \varkappa_1=\varkappa_2=\ldots=\varkappa_r=m.$$

E. DERIVATION.

We will next give the four formulas of derivation, corresponding in order to those of reduction.

1. THE FIRST FORMULA.

The first formula of derivation is

$$\begin{pmatrix} 0^{\lambda_0}\ 1^{\lambda_1}\ldots\ldots n^{\lambda_n} \\ 0^m\ \varkappa_1\ \varkappa_2\ldots\varkappa_n \end{pmatrix}=\begin{pmatrix} 0^{\lambda_0+\lambda}\ 1^{\lambda_1}\ldots\ldots n^{\lambda_n} \\ 0^{m+\lambda}\ \varkappa_1\ \varkappa_2\ldots\varkappa_n \end{pmatrix},$$

and is seen to be true in that we can multiply both the term and $b_0{}^m\Sigma$ by $b_0{}^\lambda$.

2. THE SECOND FORMULA.

By B, 1, (2),

$$\begin{pmatrix} 0^{\lambda_0}\ 1^{\lambda_1}\ldots\ldots n^{\lambda_n} \\ 0^m\ \varkappa_1\ \varkappa_2\ldots\varkappa_n \end{pmatrix}=(-1)^{mn}\begin{pmatrix} 0^{\lambda_n}\ 1^{\lambda_{n-1}}\ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_0} \\ 0^m(m-\varkappa_n)(m-\varkappa_{n-1}\ldots\ (m-\varkappa_1) \end{pmatrix}.$$

By the first formula of derivation 1, and by B, 1, (2),

$$\begin{pmatrix} 0^{\lambda_n}\ 1^{\lambda_{n-1}}\ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_0} \\ 0^m(m-\varkappa_n)(m-\varkappa_{n-1})\ldots(m-\varkappa_1) \end{pmatrix}=\begin{pmatrix} 0^{\lambda_n+\kappa}\ 1^{\lambda_{n-1}}\ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_0} \\ 0^{m+\kappa}(m-\varkappa_n)(m-\varkappa_{n-1})\ldots(m-\varkappa_1) \end{pmatrix}$$

$$=(-1)^{(m+\kappa)n}\begin{pmatrix} 0^{\lambda_0}\ 1^{\lambda_1}\ldots\ldots\ldots\ldots n^{\lambda_n+\kappa} \\ 0^{m+\kappa}(\varkappa_1+\varkappa)\ldots(\varkappa_n+\varkappa) \end{pmatrix}.$$

Therefore by substitution,

$$\begin{pmatrix} 0^{\lambda_0}\ 1^{\lambda_1}\ldots\ldots n^{\lambda_n} \\ 0_m\ \varkappa_1\ \varkappa_2\ldots\kappa_n \end{pmatrix}=(-1)^{n\kappa}\begin{pmatrix} 0^{\lambda_0}\ 1^{\lambda_1}\ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_n+\kappa} \\ 0^{m+\kappa}(\varkappa_1+\kappa)(\varkappa_2+\varkappa)\ldots(\varkappa_n+\varkappa) \end{pmatrix},$$

and this is the second formula of derivation.

3. THE THIRD FORMULA.

The third formula of derivation comes from the third formula of reduction in that we turn the latter about, and write n^1 instead of r, $n^1+\varkappa$ instead of n, and consequently $\varkappa$ instead of $n-r$. We then have, dropping accents,

$$\begin{pmatrix} 0^{\lambda_0}\ 1^{\lambda_1}\ldots\ldots n^{\lambda_n} \\ 0^m\ \varkappa_1\ \varkappa_2\ldots\varkappa_n \end{pmatrix}=\begin{pmatrix} 0^{\lambda_0}\ 1^{\lambda_1}\ldots n^{\lambda_n}(n+1)^0\ldots(n+\varkappa)^0 \\ 0^m\ \varkappa_1\ \varkappa_2\ldots\ldots\ldots\ldots\ldots\varkappa_n\ 0^\kappa \end{pmatrix}.$$

4. THE FOURTH FORMULA.

We can obtain the fourth formula of derivation from the fourth of reduction by turning the latter about. Remembering that $\varkappa_1=\varkappa_2=\ldots=\varkappa_r=m$, the fourth formula of reduction may be written

$$\begin{pmatrix} r^{\lambda_r}\ (r+1)^{\lambda_{r+1}}\ldots\ldots n^{\lambda_n} \\ 0^m\ m^r\ \varkappa_{r+1}\ldots\ldots\ \varkappa_n \end{pmatrix}=(-1)^{mr}\begin{pmatrix} 0^{\lambda_r}\ 1^{\lambda_{r+1}}\ldots\ldots(n-r)^{\lambda_n} \\ 0^m\ \varkappa_{r+1}\ldots\ldots\ldots\varkappa_n \end{pmatrix};$$

we may now put,

$$\lambda_r=\lambda_0^1 \qquad \varkappa_{r+1}=\varkappa_1^1$$

$$\lambda_{r+1}=\lambda_1^1 \qquad \varkappa_{r+2}=\varkappa_2^1$$

$$\cdots\cdots \qquad \cdots\cdots$$

$$\lambda_n=\lambda^1{}_{n-r} \qquad \varkappa_n=\varkappa^1{}_{n-r};\ \text{we get,}$$

$$\begin{pmatrix} 0^{\lambda_0^1}\ 1^{\lambda_1^1}\ldots\ldots(n-r)^{\lambda^1{}_{n-r}} \\ 0^m\ \varkappa_1^1\ \varkappa_2^1\ldots\ldots\ldots\varkappa^1{}_{n-r} \end{pmatrix}=(-1)^{mr}\begin{pmatrix} r^{\lambda_0^1}(r+1)^{\lambda_1^1}\ldots\ldots n^{\lambda^1{}_{n-r}} \\ 0^m\ m^r\varkappa_1^1\ \varkappa_2^1\ldots\ldots\ \varkappa^1{}_{n-r} \end{pmatrix}.$$

Finally putting $n-r=n^1$, $n=n^1+r$, and dropping all accents,

$$\begin{pmatrix} 0^{\lambda_0}\ 1^{\lambda_1}\ldots\ldots n^{\lambda_n} \\ 0^m\ \varkappa_1\ \varkappa_2\ldots\varkappa_n \end{pmatrix}=(-1)^{mr}\begin{pmatrix} r^{\lambda_0}(r+1)^{\lambda_1}\ldots\ldots(n+r)^{\lambda_n} \\ 0^m\ m^r\ \varkappa_1\ \varkappa_1\ldots\ldots\ldots\varkappa_n \end{pmatrix},$$

and this is the fourth formula of derivation.

5. The Four Formulas of Derivation.

We collect the four formulas of derivation. They are :

$$\begin{pmatrix} 0^{\lambda_0}\ 1^{\lambda_1}\ldots\ldots\ n^{\lambda_n} \\ 0^m\ \varkappa_1\ \varkappa_2\ldots\varkappa_n \end{pmatrix}=\begin{pmatrix} 0^{\lambda_0+\lambda}\ 1^{\lambda_1}\ldots\ldots n^{\lambda_n} \\ 0^{m+\lambda}\ \varkappa_1\ \varkappa_2\ldots\varkappa_n \end{pmatrix}$$

$$\begin{pmatrix} 0^{\lambda_0}\ 1^{\lambda_1}\ \ldots\ .n^{\lambda_n} \\ 0^m\ \varkappa_1\ \varkappa_2\ldots\varkappa_n \end{pmatrix}=(-)^{n\kappa}\begin{pmatrix} 0^{\lambda_0}\ 1^{\lambda_1}\ldots\ldots\ldots\ldots\ldots\ldots\ldots n^{\lambda_n+\kappa} \\ 0^{m+\kappa}(\varkappa_1+\varkappa)(\varkappa_2+\varkappa)\ldots(\varkappa_n+\varkappa) \end{pmatrix}$$

$$\begin{pmatrix} 0^{\lambda_0}\ 1^{\lambda_1}\ldots\ldots n^{\lambda_n} \\ 0^m\ \varkappa_1\ \varkappa_2\ldots\ .\varkappa_n \end{pmatrix}=\begin{pmatrix} 0^{\lambda_0}\ 1^{\lambda_1}\ldots n^{\lambda_n}(n+1)^0\ldots(n+\varkappa)^0 \\ 0^m\ \varkappa_1\ \varkappa_2\ \ldots\ldots\ldots\ldots\ldots\varkappa_n\ 0^{\kappa} \end{pmatrix}$$

$$\begin{pmatrix} 0^{\lambda_0}\ 1^{\lambda_1}\ldots\ldots n^{\lambda_n} \\ 0^m\ \varkappa_1\ \varkappa_2\ldots\varkappa_n \end{pmatrix}=(-1)^{mr}\begin{pmatrix} r^{\lambda_0}(r+1)^{\lambda_1}\ldots(n+r)^{\lambda_n} \\ 0^m m^r\ \varkappa_1\ \varkappa_2\ldots\ldots\ldots\varkappa_n \end{pmatrix}.$$

These four kinds of derivation here correspond to the third, first, second, and fourth kinds of derivation, respectively, in the resultant theory.

F. THE CONJUGATE PRODUCT AND FUNCTION.

1. Definitions.

With respect to the product $(a_{r_1})^{p_1}(a_{r_2})^{p_2}\ldots(a_{r_\mu})^{p_\mu}$, the function

$$a_0{}^n\Sigma(ar_1)^{p_1+p_2+\ldots\ldots p_\mu}[\alpha(r_2-r_1)]^{p_2+p_3+\ldots\ldots p_\mu}[\alpha(r_3-r_2)]^{p_3+p_4+\ldots\ldots p_\mu}\ldots$$

$$[\alpha(r_\mu-r_{\mu-1})]^{p_\mu}$$

$$=a_0{}^n\Sigma(\alpha_1\alpha_2\ldots\alpha_r)^{p_1}(\alpha_1\alpha_2\ldots\alpha_{r_2})^{p_2}(\alpha_1\alpha_2\ldots\alpha_{r_3})^{p_3}\ldots(\alpha_1\alpha_2\ldots\alpha_{r_\mu})^{p_\mu},$$

is called the conjugate symmetric function, and with respect to the function, the product is called the conjugate product.

The number n is the order of the function and $n = p_1 + p_2 + \ldots . p_\mu$. If the function is given as

$$a_0^{\kappa_1} \Sigma \alpha_1^{\kappa_1} \alpha_2^{\kappa_2} \ldots \alpha_n^{\kappa_n} = a_0^{\kappa_1} \Sigma \alpha_1^{\kappa_1 - \kappa_2} (\alpha_1 \alpha_2)^{\kappa_2 - \kappa_3} (\alpha_1 \alpha_2 \alpha_3)^{\kappa_3 - \kappa_4} \ldots (\alpha_1 \alpha_2 \ldots . \alpha_n)^{\kappa_n},$$

it is clear that the conjugate product is,

$$a_1^{\kappa_1 - \kappa_2}\, a_2^{\kappa_2 - \kappa_3}\, a_3^{\kappa_3 - \kappa_4} \ldots . a_n^{\kappa_n}.$$

2. THE COEFFICIENT OF THE CONJUGATE PRODUCT IN A SYMMETRIC FUNCTION. COMPLETELY REDUCIBLE FORMS.

It will next be proved that the coefficient of the conjugate product in a symmetric function is completely reducible, and is equal to $(-1)^w$, where w is the weight of the function. According to 1, this coefficient may be written :

$$\begin{pmatrix} 1^{\kappa_1 - \kappa_2}\, 2^{\kappa_2 - \kappa_3} \ldots . (n-1)^{\kappa_{n-1} - \kappa_n}\, n^{\kappa_n} \\ 0^{\kappa_1}\, \varkappa_1\, \varkappa_2 \ldots\ldots\ldots\ldots\ldots\ldots \varkappa_n \end{pmatrix}.$$

By D, 2 (second reduction), we may reduce this, as follows :

$$\begin{pmatrix} 1^{\kappa_1 - \kappa_2}\, 2^{\kappa_2 - \kappa_3} \ldots . (n-1)^{\kappa_{n-1} - \kappa_n}\, n^{\kappa_n} \\ 0^{\kappa_1}\, \varkappa_1\, \varkappa_2 \ldots\ldots\ldots\ldots\ldots\ldots \varkappa_n \end{pmatrix}$$

$$= (-1)^{n\kappa_n} \begin{pmatrix} 1^{\kappa_1 - \kappa_2}\, 2^{\kappa_2 - \kappa_3} \ldots\ldots\ldots . (n-1)^{\kappa_{n-1} - \kappa_n} \\ 0^{\kappa_1 - \kappa_n} (\varkappa_1 - \varkappa_n)(\varkappa_2 - \varkappa_n) \ldots . (\varkappa_{n-1} - \varkappa_n) \end{pmatrix}$$

$$= (-1)^{n\kappa_n + (n-1)(\kappa_{n-1} - \kappa_n)} \begin{pmatrix} 1^{\kappa_1 - \kappa_2}\, 2^{\kappa_2 - \kappa_3} \ldots\ldots\ldots\ldots\ldots . (n-2)^{\kappa_{n-2} - \kappa_{n-1}} \\ 0^{\kappa_1 - \kappa_{n-1}} (\varkappa_1 - \varkappa_{n-1})(\varkappa_2 - \varkappa_{n-1}) \ldots . (\varkappa_{n-2} - \varkappa_{n-1}) \end{pmatrix}.$$

We have made here two reductions, one after the other. Proceeding in this way, we can now see that after n reductions of the same kind, we obtain,

$$\begin{pmatrix} 1^{\kappa_1 - \kappa_2}\, 2^{\kappa_2 - \kappa_3} \ldots\ldots n^{\kappa_n} \\ 0^{\kappa_1}\, \varkappa_1\, \varkappa_2 \ldots\ldots\ldots \varkappa_n \end{pmatrix}$$

$$= (-1)^{n\varkappa_n + (n-1)(\varkappa_{n-1} - \varkappa_n) + (n-2)(\varkappa_{n-2} - \varkappa_{n-1}) + \ldots . (\varkappa_1 - \varkappa_2)} \binom{1^0}{0^0}$$

$$= (-1)^{\kappa_1 + \kappa_2 + \ldots . \kappa_n} \binom{1}{1} = (-1)^w.$$

This is the completely reducible form in the theory of symmetric functions. It may also be written as

$$\begin{pmatrix} 0^{\lambda_0}\, 1^{\lambda_1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . n^{\lambda_n} \\ 0^m (\lambda_1 + \lambda_2 + \ldots . \lambda_n)(\lambda_2 + \lambda_3 + \ldots . \lambda_n)(\lambda_3 + \lambda_4 + \ldots . \lambda_n) \ldots . (\lambda_{n-1} + \lambda_n)\lambda_n \end{pmatrix}.$$

3. Correspondence of the Conjugate Product with the Completely Reducible Term of $R_{m,n}$.

It will yet be shown more in detail that the conjugate product in a symmetric function corresponds to the completely reducible term in the resultant theory. According to the method of A, 5, the resultant of f and ϕ may be expressed as

$$\Sigma(-1)^{mn+g_1}(a_{r_1})^{p_1}(a_{r_2})^{p_2}\,..\,(a_{r_\mu})^{p_\mu}\,(b_0)^m\Sigma(\beta p_1)^{m-r_1}\,(\beta p_2)^{m-r_2}\,..\,(\beta p_\mu)^{m-r_\mu},$$

where g_1 is the weight of $(a_{r_1})^{p_1}(a_{r_2})^{p_2}\,....\,(a_{r_\mu})^{p_\mu}$. [$r_1$, $r_2\,....\,r_\mu$ are supposed to be arranged in the order of ascending magnitude]. A group of terms will be represented by

$$(-1)^{mn+g_1}(a_{r_1})^{p_1}(a_{r_2})^{p_2}\,....\,(a_{r_\mu})^{p_\mu}\,(b_0)^m\Sigma(\beta p_1)^{m-r_1}\,....\,(\beta p_\mu)^{m-r_\mu}.$$

Among these terms will be the one containing the conjugate product of the symmetric function involved. It will be

$$(-1)^{mn+g_1}(a_{r_1})^{p_1}(a_{r_2})^{p_2}\,....\,(a_{r_\mu})^{p_\mu}\,(b_0)^m\frac{(-1)^{mn-g_1}}{b_0{}^{m-r_1}}(b_{p_1})^{r_2-r_1}\,(b_{p_1+p_2})^{r_3-r_1}$$

$$....\,(b_{p_1+p_2+.....p_\mu})^{m-r_\mu}=(a_{r_1})^{p_1}(a_{r_2})^{p_2}\,....\,(a_{r_\mu})^{p_\mu}\,(b_0)^{r_1}(b_{p_1})^{r_2-r_1}\,(b_{p_1+p_2})^{r_3-r_1}$$

$$....\,(b_{p_1+p_2+.....p_\mu})^{m-r_\mu}.$$

But this by A, 1, (2) is the completely reducible term of $R_{m,n}$.

G. RECURRENCE FORMULA FOR THE COEFFICIENT

$$\begin{pmatrix}0^{\lambda_0}\ 1^{\lambda_1}\ 2^{\lambda_2}\,.................\,n^{\lambda_n}\\ 0^m m^{\mu_0}(m-1)^{\mu_1}(m-2)^{\mu_2}\,....\,0^{\mu_m}\end{pmatrix}.$$

1. Statement of Requirement.

Since all the coefficients of terms in symmetric functions can be reduced to such as are normal forms, or else are completely reducible and have the coefficient $(-1)^w$, we require farther only a means of calculating the normal forms. We will obtain a recurrence formula for the normal form

$$\begin{pmatrix}0^{\lambda_0}\ 1^{\lambda_1}\,...........\,n^{\lambda_n}\\ 0^m m^{\mu_0}(m-1)^{\mu_1}\,....\,0^{\mu_m}\end{pmatrix},\ \text{where } m\geqq n.$$

2. Derivation of the Formula.

From the equation $\phi=0$, we have

$$b_0\beta_1{}^m+b_1\beta_1{}^{m-1}+b_2\beta_1{}^{m-2}+\,....\,+b_r\beta_1{}^{m-r}+\,....\,+b_n\beta_1{}^{m-n}=0.$$

We multiply this equation by

$$(b_0)^{m-1}(\beta_2\beta_3\dots\beta_{\mu_0})^m(\beta_{\mu_0+1}\dots\beta_{\mu_0+\mu_1})^{m-1}\dots(\beta_{\mu_{m-2}+1}\dots\beta_{\mu_0+\mu_1+\dots\mu_{m-1}})^1$$

$(\beta_{\mu_{m-1}+1}\dots\beta_{\mu_0+\dots\dots\mu_m})^0$ and get $(b_0)^m(\beta\mu_0)^m(\beta\mu_1)^{m-1}(\beta\mu_2)^{m-2}\dots$

$$(\beta\mu_m)^0+(b_0)^{m-1}b_1[\beta(\mu_0-1)]^m[\beta(\mu_1+1)]^{m-1}\dots(\beta\mu_m)^0+\dots$$

$$+(b_0)^{m-1}b_r[\beta(\mu_0-1)]^m(\beta\mu_1)^{m-1}(\beta\mu_2)^{m-2}\dots[\beta(\mu_r+1)]^{m-r}\dots(\beta\mu_m)^0+\dots=0.$$

We will next take the sum of all such equations, using all possible substitutions in the summation. We obtain a series of symmetric functions, of which the first contains $\dfrac{m!}{\mu_0!\,\mu_1!\,\mu_2!\,\dots\,\mu_m!}$ terms, and the $(r+1)$st $\dfrac{m!}{(\mu_0-1)!\,\mu_1!\,\mu_2!\dots(\mu_r+1)!\dots\mu_m!}$ terms. These numbers can be written as $\dfrac{1}{\mu_0}\lambda$, and $\dfrac{1}{\mu_r+1}\lambda$, where $\lambda=\dfrac{m!}{(\mu_0-1)!\,\mu_1!\,\mu_2!\dots\mu_r!\dots\mu_m!}$.

But by this mode of summation, the first function will be repeated $\dfrac{n!}{\dfrac{1}{\mu_0}\lambda}=\mu_0\dfrac{n!}{\lambda}$ times, and the $(r+1)$st function $(\mu_r+1)\dfrac{n!}{\lambda}$ times.

We get therefore, by our summation, dividing by the common factor $n!/\lambda$,

$$\mu_0 b_0{}^m\Sigma(\beta\mu_0)^m(\beta\mu_1)^{m-1}(\beta\mu_2)^{m-2}\dots(\beta\mu_r)^{m-r}\dots(\beta\mu_m)^0+$$

$$(b_0)^{m-1}\sum_{r=1}^{r=n}b_r(1+\mu_r)\Sigma[\beta(\mu_0-1)]^m(\beta\mu_1)^{m-1}(\beta\mu_2)^{m-2}\dots[\beta(\mu_r+1)]^{m-r}\dots(\beta\mu_m)^0\equiv0.$$

If in this equation we pick out the coefficient of $b_0{}^{\lambda_0}b_1{}^{\lambda_1}b_2{}^{\lambda_2}\dots b_n{}^{\lambda_n}$ we get,

$$\mu_0\begin{pmatrix}0^{\lambda_0}\ 1^{\lambda_1}\ 2^{\lambda_2}\dots\dots n^{\lambda_n}\\0^m m^{\mu_0}(m-1)^{\mu_1}\dots0^{\mu_m}\end{pmatrix}=-\sum_{r=1}^{r=n}(1+\mu_r)\begin{pmatrix}0^{\lambda_0}\ 1^{\lambda_1}\ 2^{\lambda_2}\dots\dots r^{\lambda_r-1}\dots\dots n^{\lambda_n}\\0^{m-1}m^{\mu_0-1}(m-1)^{\mu_1}\dots(m-r)^{\mu_r+1}\dots0^{\mu_m}\end{pmatrix}$$

$$=-\sum_{r=1}^{r=n}(1+\mu_r)\begin{pmatrix}0^{\lambda_0+1}\ 1^{\lambda_1}\ 2^{\lambda_2}\dots\dots r^{\lambda_r-1}\dots\dots n^{\lambda_n}\\0^m\ m^{\mu_0-1}(m-1)^{\mu_1}\dots(m-r)^{\mu_r+1}\dots0^{\mu_m}\end{pmatrix}.$$

3. CORRESPONDENCE OF THE FORMULA WITH THE RECURRENCE FORMULA IN THE RESULTANT THEORY.

The preceding formula of 2 corresponds to the recurrence formula of A, 4, in the resultant theory. The identity of the two can be shown by eliminating the coefficients of the resultant from the latter formula by means of the relation [A, 5, (2)]

$$(m-\varkappa_1)(m-\varkappa_2)\ldots\ldots(m-\varkappa_n)\ \big|\ 0^{\lambda_0}\ 1^{\lambda_1}\ 2^{\lambda_2}\ldots\ldots n^{\lambda_n}$$

$$=(-1)^{\kappa_1+\kappa_2+\ldots\ldots\kappa_n}\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} & \ldots\ldots & n^{\lambda_n}\\ 0^m & \varkappa_1 & \varkappa_2\ldots\ldots & \varkappa_n\end{pmatrix}$$

after suitably changing the notation, or we may first write the formula of 2 as

$$c_0\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} & \ldots\ldots & n^{\lambda_n}\\ 0^m & \varkappa_1 & \varkappa_2\ldots\ldots & \varkappa_n\end{pmatrix}=-\sum_{r=1}^{r=n}(1+c_r)\begin{pmatrix}0^{\lambda_0+1} & 1^{\lambda_1} & \ldots\ldots\ldots & n^{\lambda_n}\\ 0^m & (\varkappa_1-r) & \varkappa_2\ldots\ldots & \varkappa_n\end{pmatrix},$$

where at least $\varkappa_1=m$, $\varkappa_n=0$, and c_r is the number of β's which have the exponent $m-r$.

4. THE CASE WHERE $n>m$.

In case $n>m$ we may use one of the relations

$$\begin{pmatrix}0^{\lambda_0} & 1^{\lambda_1} & \ldots\ldots & n^{\lambda_n}\\ 0^m & \varkappa_1 & \varkappa_2\ldots\ldots & \varkappa_n\end{pmatrix}=\begin{pmatrix}\varkappa_1 & \varkappa_2 & \ldots\ldots\ldots & \varkappa_n\\ 0^n & 0^{\lambda_0} & 1^{\lambda_1}\ldots\ldots & n^{\lambda_n}\end{pmatrix}$$

$$=(-1)^{mn}\begin{pmatrix}(m-\varkappa_1)(m-\varkappa_2)\ldots\ldots(m-\varkappa_n)\\ 0^n0^{\lambda_n}1^{\lambda_{n-1}}\ldots\ldots\ \ldots\ldots\ldots\ldots n^0\end{pmatrix}$$

of B, 1, (2), and then compute as before by the reduction formula of 2.

[To be Concluded.]